CHAPTER 2

MODELING OF DYNAMIC SYSTEMS

PROBLEMS

Problems for Section 2-1

2-1. Find the equation of the motion of the mass-spring system shown in Fig. 2P-1. Also calculate the natural frequency of the system.

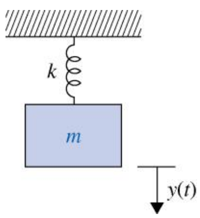
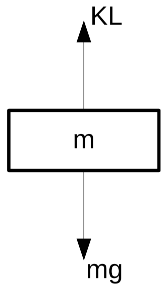


Figure 2P-1

When the mass is added to spring, then the spring will stretch from position O to position L.



The total potential energy is:

$$U_s = \frac{1}{2}K\left(L+y\right)^2$$

where y is a displacement from equilibrium position L.

The gravitational energy is:

$U_g = -mgy$

The kinetic energy of the mass-spring system is calculated by:

$$T = \frac{1}{2} m\dot{y}^2$$

As we know that $T + U_g + U_s = \text{constant}$, then

$$\frac{1}{2} m\dot{y}^2 - mgy + \frac{1}{2} K\left(L + y\right)^2 = constant$$

By differentiating from above equation, we have:

$$m\dot{y}\ddot{y} - mg\dot{y} + \dot{y}K\left(L + y\right) = 0$$

$$m\dot{y}\ddot{y} + Ky\dot{y} + \dot{y}\left(KL - mg\right) = 0$$

since $KL = mg$, therefore:

$$\dot{y}\left(m\ddot{y} + Ky\right) = 0$$

As $\dot{y}$ cannot be zero because of vibration, then $m\ddot{y} + Ky = 0$

Alternative using Newton's Law:

At equilibrium from the Free-body-diagram we have

$$KL = mg$$

Measured from equilibrium position using y(*t*), we have

$$F = Ma$$

$$-Ky\left(t\right) = M\ddot{y}\left(t\right)$$

$$or$$

$$M\ddot{y}\left(t\right) + Ky\left(t\right) = 0$$

2-2. Find its single spring-mass equivalent in the five-spring one-mass system shown in Fig. 2P-2. Also calculate the natural frequency of the system.

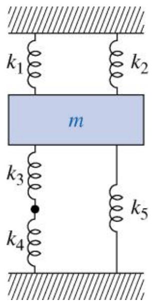
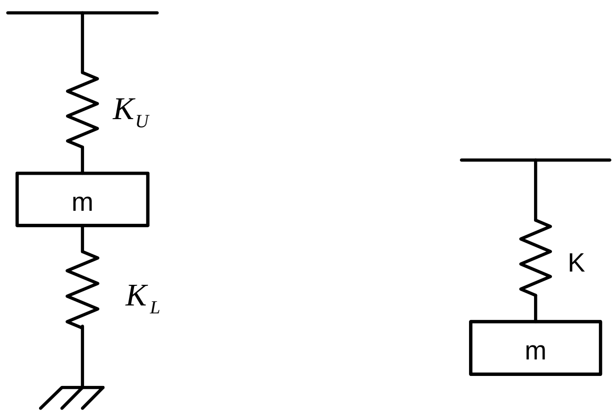


Figure 2P-2



$$K_u = K_1 + K_2$$

$$K_L = \frac{K_3 K_4}{K_3 + K_4} + K_5$$

$$\begin{aligned} K &= K_u + K_L \\ &= K_1 + K_2 + K_5 + \frac{K_3 K_4}{K_3 + K_4} \\ &= \frac{K_1 K_3 + K_2 K_3 + K_5 K_3 + K_1 K_4 + K_2 K_4 + K_4 K_5 + K_3 K_4}{K_3 + K_4} \end{aligned}$$

$$\omega_n = \sqrt{\frac{K}{m}}$$

2-3. Find the equation of the motion for a simple model of a vehicle suspension system hitting a bump. As shown in Fig. 2P-3 the mass of wheel and its mass moment of inertia are *m* and *J*, respectively. Also calculate the natural frequency of the system.

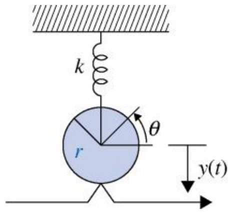


Figure 2P-3

a) **Rotational kinetic energy:** $T_{rot} = \frac{1}{2} J\dot{\theta}^2$

Translational kinetic energy: $T_T = \frac{1}{2} m\dot{y}^2$

Relation between translational displacement and rotational displacement:

$y = r\theta$

$\dot{y} = r\dot{\theta}$

$T_{Rot} = \frac{1}{2}\frac{J}{r^2}\dot{y}^2$

Potential energy: $U = \frac{1}{2}Ky^2$

From conservation of energy $T_{Rot} + T_T + U = constant$, then:

$$\frac{1}{2}\frac{J}{r^2}\dot{y}^2 + \frac{1}{2}\ m\dot{y}^2 + \frac{1}{2}\ Ky^2 = constant$$

By differentiating, we have:

$$\frac{J}{r^2}\dot{y}\ddot{y} + m\dot{y}\ddot{y} + Ky\dot{y} = 0$$

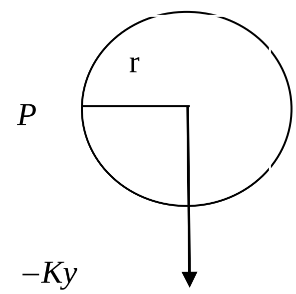
$$\dot{y}\left(\frac{J}{r^2}\ddot{y} + m\ddot{y} + Ky\right) = 0$$

Since $\dot{y}$ cannot be zero, then $J\frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$

Alternatively using Newton's law, take a moment about point *P*, assuming motion is counterclockwise, and as the wheel goes above the bump, *y* is upwards. Also we assume the system starts from equilibrium (in the vertical direction) where the spring force and the weight of the system cancel each other. So *m*g does not appear in the equations. As the wheel moves up the spring compresses by *y* measured from the equilibrium.

Assuming positive direction is counterclockwise, we have

$$\sum Mom_P = -Kyr = J\ddot{\theta} + mr\ddot{y}$$



$$J\frac{\ddot{y}}{r^2} + m\ddot{y} + Ky = 0$$

b)

Natural frequency is the coefficient of *y* divided by the coefficient of $\ddot{y}$

$$\omega_n = \sqrt{\frac{K}{m + \frac{J}{r^2}}} = r\sqrt{\frac{K}{mr^2 + J}}$$

2-4. Write the force equations of the linear translational systems shown in Fig. 2P-4.

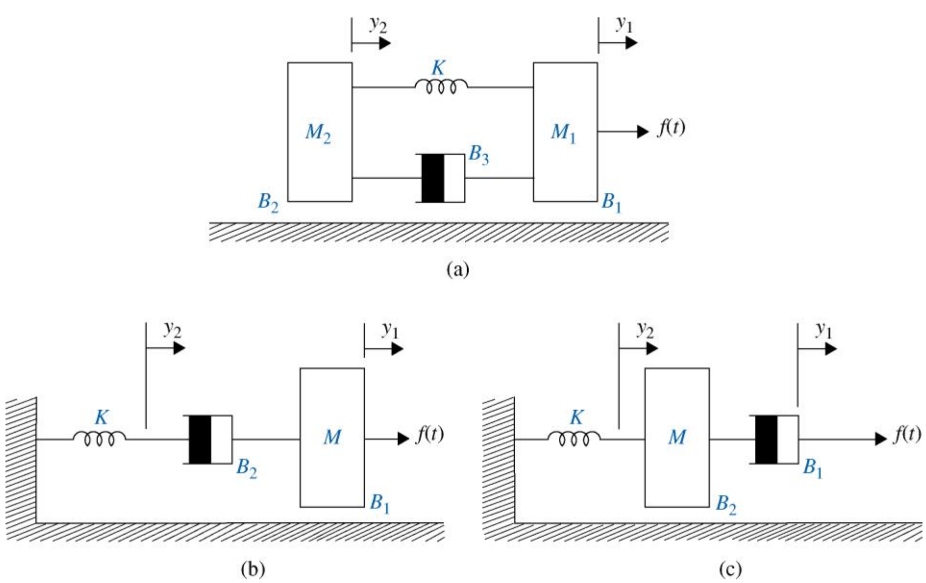


Figure 2P-4

(a) Force equations: